

Finite-dimensional global attractors for parabolic nonlinear equations with state-dependent delay

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Abstract

We deal with a class of parabolic nonlinear evolution equations with state-dependent delay. This class covers several important PDE models arising in biology. We first prove well-posedness in a certain space of functions which are Lipschitz in time. We show that the model considered generates an evolution operator semigroup S_t on a space \mathcal{C} of Lipschitz type functions over delay time interval. The operators S_t are closed for all $t \geq 0$ and continuous for t large enough. Our main result shows that the semigroup S_t possesses compact global and exponential attractors of finite fractal dimension. Our argument is based on the recently developed method of quasi-stability estimates and involves some extension of the theory of global attractors for the case of closed evolutions.

Keywords: parabolic evolution equations, state-dependent delay, global attractor, finite-dimension, exponential attractor.

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1 Introduction

Differential equations with different types of delay attract much attention during last decades. Including delay terms in differential equations is a natural step of taking into account that the majority of real-world problems depends on the pre-history of the evolution. Delay terms in an equation reflect a well-understood phenomenon that evolution of a state of a system depends not

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only on this state but rather on the states during some previous interval of time (memory of the system). This leads to infinite-dimensional dynamics even in the case of ordinary differential equations. The general theory of delay differential equations was initially developed for the simplest case of constant delays. We cite just classical monographs [2, 13, 18] on ordinary differential equations (ODEs) and milestone articles [16, 37] on partial differential equations (PDEs) with constant delays. On the other hand it is clear that the constancy of the delay is just an extra assumption made to simplify the study, but it is not really well-motivated by real-world models. To describe a process more naturally a new class of *state-dependent delay* models was introduced and intensively studied during last decades. We mention works on ODEs [14, 20, 22, 38] and on PDEs [12, 29, 30, 32, 33] with state-dependent delays.

The simplest case of a state-dependent delay is a delay explicitly given by a real-valued function $\eta : \mathbb{R} \rightarrow \mathbb{R}_+$ which depends on the value $x(t)$ at the reference time t but not on previous values of the solution $\{x(\tau), \tau \leq t\}$. This leads to terms of the form $f(x(t - \eta(x(t))))$ in the model considered. Even in this case the non-uniqueness could appear (see the scalar ODE example constructed by R.Driver [14] in 1963 for initial data from the space of continuous functions on the delay interval). The standard way for general models to avoid non-uniqueness in the case of infinite-dimensional dynamics is to consider smoother (narrower) classes of solutions. However in this case the existence problem may become critical. The main task is to find a good balance between these two issues.

In this paper we deal with a certain abstract parabolic problem with the state dependent delay term of a rather general structure. Our considerations are motivated by several biological models, see the discussion and the references in [3], [17] and [33]. Our main goal in this paper is to find appropriate phase spaces in which we can establish the well-posedness of our model and study its long time (qualitative) dynamics.

Our first result (Theorem 3.3) states well-posedness of the problem and allows us to define an evolution semigroup S_t of closed mappings on a certain Banach space of functions on the delay time interval with values in an appropriate Hilbert space. In some sense this result extends the well-posedness statements in [30, 32, 33] to more general delay terms. The main result of the paper (Theorem 4.2) states the existence of a global *finite-dimensional* attractor, the object which is responsible for long-time dynamics. We also show that the model possesses an exponential fractal global attractor (see the definition in the Appendix).

Although for some parabolic problems with state-dependent delay the existence of compact global attractors was established earlier in [30, 33], to the best of our knowledge, results on *finite-dimensional* behavior for parabolic state-dependent delay problems were not known before. The main difficulty is related to the fact the corresponding delay term is not Lipschitz on the natural energy balance space. We also mention that our Theorem 4.2 can be applied in the situation considered in [33] and gives the finite-dimensionality of the global attractor constructed in that

paper.

We note that the evolution operators S_t we construct are not continuous mapping on the phase space for t small enough. Therefore to prove the existence of a compact global attractor we use the extension of the standard theory suggested in [28]. As for dimension issues we apply the idea of the method of quasi-stability estimates developed earlier in [6, 7, 8, 9] for the second order in time evolution models which generate continuous evolution semigroups. This is possible in our case due to the continuity of evolution operator for large times. We note that in the delay case the quasi-stability method was applied earlier in [10, 11, 12] for second order models, see also [5, Chapter 6].

2 Model description

We deal with well-posedness and long-time dynamics of abstract evolution equations with delay of the form

$$\dot{u}(t) + Au(t) + F(u_t) + G(u(t)) = h, \quad t > 0, \quad (1)$$

in some Hilbert space H . Here the dot over an element means time derivative, A is linear and F, G are nonlinear operators, $h \in H$. The term $F(u_t)$ represents (nonlinear) delay effect in the dynamics. As usually for delay equations, the history segment (the state) is denoted by $u_t \equiv u_t(\theta) \equiv u(t + \theta)$ for $\theta \in [-r, 0]$.

Assumption 2.1 (Basic Hypotheses) In our study we assume that:

- (A) A is a positive operator with a discrete spectrum in a separable Hilbert space H with a dense domain $D(A) \subset H$. Hence there exists an orthonormal basis $\{e_k\}$ of H such that

$$Ae_k = \lambda_k e_k, \quad \text{with } 0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lim_{k \rightarrow \infty} \lambda_k = \infty.$$

We define the spaces H_α which are $D(A^\alpha)$ for $\alpha \geq 0$ (the domain of A^α) and the completions of H with respect to the norm $\|A^\alpha \cdot\|$ when $\alpha < 0$ (see, e.g., [25]). Here and below, $\|\cdot\|$ is the norm of H , and $\langle \cdot, \cdot \rangle$ is the corresponding scalar product. For $r > 0$, we denote for short $C_\alpha = C([-r, 0]; H_\alpha)$ which is a Banach space with the norm

$$|v|_{C_\alpha} \equiv \sup\{\|v(\theta)\|_\alpha : \theta \in [-r, 0]\},$$

where $\|v\|_\alpha = \|A^\alpha v\|$ is the norm in H_α for $\alpha \in \mathbb{R}$. We also write $C = C_0$.

- (F) The delay term $F(u_t)$ has the form $F(u_t) \equiv F_0(u(t - \eta(u_t)))$, where (a) $F_0 : H_\alpha \mapsto H_\alpha$ is globally Lipschitz for $\alpha = 0$ and $\alpha = -1/2$, i.e., there exists $L_F > 0$ such that

$$\|F_0(v) - F_0(u)\|_\alpha \leq L_F \|v - u\|_\alpha, \quad v, u \in H_\alpha, \quad \alpha = 0, -1/2; \quad (2)$$

and **(b)** $\eta : C \equiv C([-r, 0]; H) \mapsto [0, r] \subset \mathbb{R}$ is globally Lipschitz:

$$|\eta(\phi) - \eta(\psi)| \leq L_\eta |\phi - \psi|_C, \quad \phi, \psi \in C([-r, 0]; H). \quad (3)$$

(G) $G : H_{1/2} \mapsto H$ is locally Lipschitz, i.e.

$$\|G(v) - G(u)\| \leq L_G(R) \|v - u\|_{1/2}, \quad v, u \in H_{1/2}, \quad \|v\|_{1/2}, \|u\|_{1/2} \leq R, \quad (4)$$

where $L_G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nondecreasing function. We also assume that G is a potential mapping, the latter means that there exists a (Frechét differentiable) functional $\Pi(u) : H_{1/2} \rightarrow \mathbb{R}$ such that $G(u) = \Pi'(u)$ in the sense

$$\lim_{\|v\|_{1/2} \rightarrow 0} \|v\|_{1/2}^{-1} [\Pi(u+v) - \Pi(u) + \langle (u), v \rangle] = 0.$$

Moreover, we assume that **(a)** there exist positive constants c_1 and c_2 such that

$$\langle G(u), Au \rangle \geq -c_1 \|A^{\frac{1}{2}} u\|^2 - c_2, \quad u \in D(A); \quad (5)$$

and **(b)** there exist $\delta > 0$ and $m \geq 0$ such that $G : H_{1/2-\delta} \mapsto H_{-m}$ is continuous.

Our main motivating example of a system with discrete state-dependent delay is the following one:

$$\frac{\partial}{\partial t} u(t, x) - \Delta u(t, x) + b([Bu(t - \eta(u_t), \cdot)](x)) + g(u(t, x)) = h(x), \quad x \in \Omega, \quad t > 0, \quad (6)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $B : L^2(\Omega) \rightarrow L^2(\Omega)$ is a bounded operator and $b : \mathbb{R} \rightarrow \mathbb{R}$ stands for a Lipschitz map. The function $\eta : C([-r, 0]; L^2(\Omega)) \rightarrow [0, r] \subset \mathbb{R}_+$ denotes a *state-dependent discrete delay*. The Nemytskii operator $u \mapsto g(u)$ with C^1 function g represents a nonlinear non-delayed reaction term and $h(x)$ describes sources. The form of the delay term is motivated by models in population dynamics where function b is a birth function (could be $b(s) = c_1 s \cdot e^{-c_2 s}$, with $c_1, c_2 > 0$) and the delay η represents the maturity age. For more detailed discussion and further examples (the diffusive Nicholson blowflies equation, Mackey-Glass equation - the diffusive model of Hematopoiesis - blood cell production, the Lasota-Ważewska-Czyżewska model in hematology) with state-dependent delay we refer to [17] and [33] and to the references therein. We note that several special cases of the model in (6) were studied in [30, 31, 32, 33]). For instance it was assumed in [33] that $g(s) \equiv 0$, $b(s)$ is a bounded function, and B is an integral compact linear operator. This leads to nonlocal (in space) models. Our assumptions covers the non-compact case. We can take $b(s) = s$ and $B = Id$, for instance. We also note that if we equip (6) with the Dirichlet boundary condition, then the dissipativity property in (5) holds provided $g \in C^1(\mathbb{R})$, $g(0) = 0$ and the derivative $g'(s)$ is bounded from below. This follows by the standard integration by parts. Thus population dynamics models with nonlinear sink/source feedback terms

can be included in consideration. For this kind of biological models, but with state-*independent* delay, we refer to [39].

We equip the equation (1) with the initial condition

$$u(\theta) = \varphi(\theta), \quad \theta \in [-r, 0], \quad (7)$$

and for initial data φ consider the space

$$\mathcal{CL} \equiv \left\{ \varphi \in C([-r, 0]; H) \mid \text{Lip}_{[-r, 0]}(A^{-\frac{1}{2}}\varphi) < +\infty; \varphi(0) \in D(A^{\frac{1}{2}}) \right\}, \quad (8)$$

where

$$\text{Lip}_{[a, b]}(\varphi) \equiv \sup_{s \neq t} \left\{ \frac{\|\varphi(s) - \varphi(t)\|}{|s - t|} : s, t \in [a, b], s \neq t \right\}$$

denotes the corresponding Lipschitz constant. One can show that the space \mathcal{CL} consists of continuous functions φ on $[-r, 0]$ with values in H such that $\varphi(0) \in H_{1/2}$ and which are absolutely continuous in $H_{-1/2}$. The latter means that there exists the derivative $\dot{\varphi} \in L^\infty(-r, 0; H_{-1/2})$ such that

$$\varphi(s) = \varphi(0) - \int_s^0 \dot{\varphi}(\xi) d\xi, \quad s \in [-r, 0],$$

and

$$\text{Lip}_{[-r, 0]}(A^{-\frac{1}{2}}\varphi) = \text{ess sup} \left\{ \|A^{-\frac{1}{2}}\dot{\varphi}(s)\| : s \in [-r, 0] \right\} \equiv |\dot{\varphi}|_{L^\infty(-r, 0; H_{-1/2})}.$$

We equip the space \mathcal{CL} with the natural norm

$$|\varphi|_{\mathcal{CL}} \equiv \max_{s \in [-r, 0]} \|\varphi(s)\| + \text{Lip}_{[-r, 0]}(A^{-\frac{1}{2}}\varphi) + \|A^{\frac{1}{2}}\varphi(0)\|. \quad (9)$$

We note that the delay term $F(\varphi) \equiv F_0(\varphi(-\eta(\varphi)))$ in (1) is well-defined for every $\varphi \in C$ and possesses the property (see (2) for $\alpha = 0$) $\|F(\varphi) - F(0)\| \leq L_F \|\varphi(-\eta(\varphi))\| \leq L_F |\varphi|_C$, hence

$$\|F(\varphi)\| \leq c_1 + c_2 |\varphi|_C, \quad \varphi \in C, \quad (10)$$

with $c_1 = \|F(0)\|$ and $c_2 = L_F$. However it is not Lipschitz on the space C . One can only show that the delay term F satisfies the inequality

$$\|F(\varphi) - F(\psi)\|_{-1/2} \leq L_F \left(1 + \text{Lip}_{[-r, 0]}(A^{-\frac{1}{2}}\varphi) \right) |\varphi - \psi|_C \quad (11)$$

for every $\varphi \in \mathcal{CL}$ and $\psi \in C$. Using the terminology of [26] we can call this mapping F “almost Lipschitz” from C into $H_{-1/2}$, see also discussion in [20].

Remark 2.2 We can also include in (1) a delay term $M(u_t)$ which is defined by a globally Lipschitz function from $C(-r, 0; H_{1/2})$ into H . We will not pursue this generalization because our main goal is state-dependent delay models.

3 Well-posedness

In this section we prove the existence and uniqueness theorem and study properties of solutions. Then we use these results to construct the corresponding evolution semigroup and describe its dynamical properties.

We introduce the following definition.

Definition 3.1 (Strong solution) A vector-function

$$u(t) \in C([-r, T]; H) \cap C([0, T]; H_{1/2}) \cap L^2(0, T; H_1) \quad (12)$$

is said to be a (strong) solution to the problem defined by (1) and (7) on $[0, T]$ if

(a) $u(\theta) = \varphi(\theta)$ for $\theta \in [-r, 0]$;

(b) $\forall v \in L^2(0, T; H)$ such that $\dot{v} \in L^2(0, T; H_{-1})$ and $v(T) = 0$ we have that

$$\begin{aligned} - \int_0^T \langle u(t), \dot{v}(t) \rangle dt + \int_0^T \langle Au(t), v(t) \rangle dt \\ + \int_0^T \langle F(u_t) + G(u(t)), v(t) \rangle dt = \langle \varphi(0), v(0) \rangle + \int_0^T \langle h, v(t) \rangle dt. \end{aligned} \quad (13)$$

Remark 3.2 Let $u(t)$ be a strong solution on an interval $[0, T]$ with some $\varphi \in C$. Then it follows from (12) and also from (4) and (10) that

$$F(u_t) + G(u(t)) - h \in L^\infty(0, T; H).$$

This allows us to conclude from (12) and (13) that

$$\dot{u}(t) \in L^\infty(0, T; H_{-1/2}) \cap L^2(0, T; H). \quad (14)$$

Moreover, the relation in (13) implies that $u(t)$ satisfies (1) for almost all $t \in [0, T]$ as an equality in H . We also note that relations (12) and (14) yield

$$u_t \in \mathcal{CL} \text{ for every } t \in [0, T] \text{ and } \max_{[0, T]} |u_t|_{\mathcal{CL}} < +\infty \quad (15)$$

for every strong solution u with initial data φ from the space \mathcal{CL} defined in (8).

Our first result is the following theorem on the existence and uniqueness of solutions.

Theorem 3.3 *Let Assumption 2.1 be in force. Assume that $\varphi \in \mathcal{CL}$, see (8). Then the initial-value problem defined by (1) and (7) has a unique strong solution on any time interval $[0, T]$. This solution possesses the property*

$$\dot{u}(t) \in C(0, T; H_{-1/2}) \cap L^2(0, T; H) \quad (16)$$

and satisfies the estimate

$$\|A^{-1/2}\dot{u}(t)\|^2 + \|A^{1/2}u(t)\|^2 + \int_0^t [\|\dot{u}(\tau)\|^2 + \|Au(\tau)\|^2] d\tau \leq C_T(R) \quad (17)$$

for all $t \in [0, T]$ and $\|A^{1/2}\varphi(0)\|^2 + |\varphi|_C^2 \leq R^2$. Moreover, for every two strong solutions u^1 and u^2 with initial data φ^1 and φ^2 from \mathcal{CL} we have that

$$\sup_{\tau \in [0, t]} \|u^1(\tau) - u^2(\tau)\|^2 + \int_0^t \|A^{1/2}(u^1(\tau) - u^2(\tau))\|^2 \leq C_R(T) |\varphi^1 - \varphi^2|_C^2, \quad \forall t \in [0, T], \quad (18)$$

for all φ^i such that $|\varphi^i|_{\mathcal{CL}} \leq R$.

Proof. To prove the existence we use the standard compactness method [24] based on Galerkin approximations with respect to the eigen-basis $\{e_k\}$ of the operator A (see Assumption 2.1 (A)).

We define a Galerkin approximate solution of order m by the formula

$$u^m = u^m(t, x) = \sum_{k=1}^m g_{k,m}(t) e_k,$$

where the functions $g_{k,m}$ are defined on $[-r, T]$, absolutely continuous on $[0, T]$ and such that the following equations are satisfied

$$\begin{cases} \langle \dot{u}^m + Au^m + F(u_t^m) + G(u^m) - h, e_k \rangle = 0, & t > 0, \\ \langle u^m(\theta), e_k \rangle = \langle \varphi(\theta), e_k \rangle, & \forall \theta \in [-r, 0], \quad \forall k = 1, \dots, m. \end{cases} \quad (19)$$

The equation in (19) is a system of (ordinary) differential equations in \mathbb{R}^m with a concentrated (discrete) state-dependent delay for the unknown vector function $U(t) \equiv (g_{1,m}(t), \dots, g_{m,m}(t))$ (for the corresponding theory see [38] and also the survey [20]).

The condition $\varphi \in \mathcal{CL}$ implies that the function $U(\cdot)|_{[-r, 0]} \equiv P_m \varphi(\cdot)$, which defines initial data, is Lipschitz continuous as a function from $[-r, 0]$ to \mathbb{R}^m . Here P_m is the orthogonal projection onto the subspace $\text{Span}\{e_1, \dots, e_m\}$. Hence, we can apply the theory of ODEs with discrete state-dependent delay (see e.g. [20]) to get the *local* existence of solutions to (19).

Next, we derive an a priori estimate which allows us to extend solutions u^m to (19) on an arbitrary time interval $[0, T]$. We also use it for the compactness of the set of approximate solutions.

We multiply the first equation in (19) by $\lambda_k g_{k,m}$ and sum for $k = 1, \dots, m$ to get

$$\frac{1}{2} \frac{d}{dt} \|A^{1/2}u^m(t)\|^2 + \|Au^m(t)\|^2 + \langle F(u_t^m) + G(u^m(t)) - h, Au^m(t) \rangle = 0.$$

Due to (10) and (5) this implies that

$$\begin{aligned} \frac{d}{dt} \left[\|A^{1/2}u^m(t)\|^2 + \int_0^t \|Au^m(\tau)\|^2 d\tau \right] &\leq c[1 + |u_t^m|_C^2 + \|A^{1/2}u^m(t)\|^2] \\ &\leq c_0[1 + |\varphi|_C^2] + c_1 \max_{\tau \in [0, t]} \|A^{1/2}u^m(\tau)\|^2 \end{aligned}$$

Integrating the last inequality we can easily see that the function

$$\Psi(t) = \max_{\tau \in [0, t]} \|A^{1/2}u^m(\tau)\|^2 + \int_0^t \|Au^m(\tau)\|^2 d\tau$$

satisfies the inequality

$$\Psi(t) \leq 2\|A^{1/2}\varphi(0)\|^2 + 2tc_0[1 + |\varphi|_C^2] + 2c_1 \int_0^t \Psi(\tau) d\tau.$$

Therefore Gronwall's lemma gives us the a priori estimate

$$\|A^{1/2}u^m(t)\|^2 + \int_0^t \|Au^m(\tau)\|^2 d\tau \leq 2e^{at} \left[\|A^{1/2}\varphi(0)\|^2 + bt[1 + |\varphi|_C^2] \right], \quad (20)$$

for all t from an existence interval, where a and b are positive constants. This a priori estimate allows us to extend approximate solutions on every time interval $[0, T]$ such that (20) remains true for every $t \in [0, T]$.

Now we establish additional a priori bounds. Using (20), (4) and (10) from the first equation in (19) we obtain that

$$\|\dot{u}^m(t) + Au^m(t)\| \leq \|F(u_t^m)\| + \|G(u^m(t))\| + \|h\| \leq C(R, T), \quad t \in [0, T],$$

provided $\|A^{1/2}\varphi(0)\|^2 + |\varphi|_C^2 \leq R^2$. Thus by (20) we obtain the estimate

$$\|A^{1/2}u^m(t)\|^2 + \int_0^t [\|\dot{u}^m(t)\|^2 + \|Au^m(\tau)\|^2] d\tau \leq C_T(R) \quad (21)$$

for all $t \in [0, T]$ and $\|A^{1/2}\varphi(0)\|^2 + |\varphi|_C^2 \leq R^2$. It also follows from (19) that

$$\sup_{t \in [0, T]} \|A^{-1/2}\dot{u}^m(t)\|^2 \leq C_T(R). \quad (22)$$

Thus

$$\{u^m\}_{m=1}^\infty \text{ is a bounded set in } W_1 \equiv L^\infty(0, T; H_{1/2}) \cap L^2(0, T; D(A)).$$

and

$$\{\dot{u}^m\}_{m=1}^\infty \text{ is a bounded set in } W_2 \equiv L^\infty(0, T; H_{-1/2}) \cap L^2(0, T; H).$$

Hence, there exist a subsequence $\{(u^k; \dot{u}^k)\}$ and an element $(u; \dot{u}) \in Z_1 \equiv W_1 \times W_2$ such that

$$\{(u^k; \dot{u}^k)\} \text{ *weak converges to } (u; \dot{u}) \text{ in } Z_1.$$

By the Aubin-Dubinski theorem [34, Corollary 4] we also have

$$u^m \rightarrow u \text{ in } C(0, T; H_{1/2-\delta}) \cap L^2(0, T; H_{1-\delta}).$$

Now the proof that any *-weak limit $u(t)$ is a solution is standard. To make the limit transition in the nonlinear terms F and G we use relation (11) and Assumption 2.1(**Gb**).

The property $u(t) \in C([0, T]; H_{1/2})$ follows from the well-known continuous embedding (see also [25, Theorem 1.3.1] or [35, Proposition 1.2]):

$$\{u \in L^2(0, T; H_1) : \dot{u} \in L^2(0, T; H)\} \subset C([0, T]; H_{1/2}).$$

The continuity of \dot{u} in $H_{-1/2}$ follows from equation (1) and from continuity of u in $H_{1/2}$. Thus the existence of strong solutions is proved. It is easy to see from (21) and (22) that the strong solution constructed satisfies (17).

Now we use this fact to prove the uniqueness.

Let u^1 and u^2 be two solutions (at this point we do not assume that they have the same initial data). Then the difference $z = u^1 - u^2 \in C([0, T]; H_{1/2}) \cap L^2(0, T; H_1)$ is a strong solution to the linear parabolic type (non-delay) equation

$$\dot{z}(t) + Az(t) = f(t), \quad t > 0, \quad \text{with} \quad f(t) \equiv F(u_t^2) - F(u_t^1) + G(u^2(t)) - G(u^1(t)). \quad (23)$$

By Remark 3.2 $f \in L^\infty(0, T; H)$. From (4) and (11) using (15) we also have that

$$\|G(u^2(t)) - G(u^1(t))\| \leq L(\varrho)\|z(t)\|_{1/2}, \quad t \in [0, T],$$

and

$$\|A^{-1/2}(F(u_t^2) - F(u_t^1))\| \leq L_F(1 + \varrho)|z_t|_C, \quad t \in [0, T],$$

for every $\varrho \geq \max_{[0, T]} \{|u_t^1|_{\mathcal{CL}} + |u_t^2|_{\mathcal{CL}}\}$. Therefore

$$\begin{aligned} |\langle f(t), z(t) \rangle| &\leq L_F(1 + \varrho)|z_t|_C\|z(t)\|_{1/2} + L(\varrho)\|z(t)\|_{1/2}\|z(t)\| \\ &\leq \frac{1}{2}\|z(t)\|_{1/2}^2 + C(\varrho)|z_t|_C. \end{aligned}$$

Thus using the standard multiplier z in (23) we obtain that

$$\frac{d}{dt}\|z(t)\|^2 + \|A^{1/2}z\|^2 \leq C(\varrho)\|z_t\|_C^2 \leq C(\varrho) \left[|\varphi^1 - \varphi^2|_C^2 + \sup_{\tau \in [0, t]} \|z(\tau)\|_C^2 \right]$$

for every $\varrho \geq \max_{[0, T]} \{|u_t^1|_{\mathcal{CL}} + |u_t^2|_{\mathcal{CL}}\}$. Applying Gronwall's lemma we obtain

$$\sup_{\tau \in [0, t]} \|u^1(\tau) - u^2(\tau)\|^2 + \int_0^t \|A^{1/2}(u^1(\tau) - u^2(\tau))\|^2 \leq C(\varrho)|\varphi^1 - \varphi^2|_C^2, \quad \forall t \in [0, T], \quad (24)$$

This implies uniqueness of strong solutions.

As a by-product the uniqueness yields that *any* strong solution satisfies (17). Therefore we can apply (24) with $\varrho > R + C_T(R)$ to obtain (18).

Thus the proof of Theorem 3.3 is complete. \square

Theorem 3.3 allows us to define an evolution semigroup S_t on the space \mathcal{CL} (see (8)) by the formula

$$S_t\varphi \equiv u_t, \quad t \geq 0, \quad (25)$$

where $u(t)$ is the unique solution to the problem (1) and (7). We note that (18) implies that S_t is *almost* locally Lipschitz on C , i.e.,

$$|S_t\varphi^1 - S_t\varphi^2|_C \leq C_R(T)|\varphi^1 - \varphi^2|_C \text{ for every } \varphi^i \in \mathcal{CL}, \quad |\varphi^i|_{\mathcal{CL}} \leq R, \quad t \in [0, T]$$

However, it seems that a similar bound is not true in the space \mathcal{CL} . We can only guarantee that $\varphi \mapsto S_t\varphi$ is a continuous mapping on \mathcal{CL} for all $t > r$. Moreover, the following assertion shows that the mapping $\varphi \mapsto S_t\varphi$ is even $\frac{1}{2}$ -Hölder on \mathcal{CL} with respect to φ when $t > r$.

Proposition 3.4 (Dependence on initial data in the space \mathcal{CL}) *Assume that the hypotheses of Theorem 3.3 are in force. Let u^1 and u^2 be two solutions on $[0, T]$ with initial data φ^1 and φ^2 from \mathcal{CL} . Then the difference $z = u^1 - u^2$ satisfies the estimate*

$$(t-r) \left[\|A^{-1/2}\dot{z}(t)\|^2 + \|A^{1/2}z(t)\|^2 \right] + \int_r^t (\tau-r) \left[\|\dot{z}(\tau)\|^2 + \|Az(\tau)\|^2 \right] d\tau \leq C_T(R)|\varphi^1 - \varphi^2|_C \quad (26)$$

for all $t \in [r, T]$ and for all initial data φ^i such that $|\varphi^i|_{\mathcal{CL}} \leq R$. This implies that for every $t > r$ the evolution semigroup S_t is $\frac{1}{2}$ -Hölder continuous in the norm of \mathcal{CL} . In the case when $t \in (0, r]$ we can guarantee the closeness of the evolution operator S_t only. This means¹ (see, e.g., [28]) that the properties $\varphi_n \rightarrow \varphi$ and $S_t\varphi_n \rightarrow \psi$ in the norm of \mathcal{CL} as $n \rightarrow \infty$ imply that $S_t\varphi = \psi$.

Proof. Multiplying (23) by Az and using (17) and (4) we obtain that

$$\frac{d}{dt} \|A^{1/2}z(t)\|^2 + \|Az(t)\|^2 \leq C \|F(u_t^2) - F(u_t^1)\|^2 + C_R(T) \|A^{1/2}z(t)\|^2, \quad t > 0.$$

From (17), (2) and (3) we also have that

$$\begin{aligned} \|F(u_t^2) - F(u_t^1)\|^2 &\leq L_F \left| \int_{t-\eta(u_t^1)}^{t-\eta(u_t^2)} \|\dot{u}^2(\xi)\| d\xi + |u_t^2 - u_t^1|_C \right|^2 \\ &\leq 2L_F \left[|\eta(u_t^1) - \eta(u_t^2)| \int_{t-r}^t \|\dot{u}^2(\xi)\|^2 d\xi + |u_t^2 - u_t^1|_C^2 \right] \leq C_T(R) |u_t^2 - u_t^1|_C \end{aligned} \quad (27)$$

for every $t \geq r$. Therefore

$$\frac{d}{dt} \|A^{1/2}z(t)\|^2 + \|Az(t)\|^2 \leq C_T(R) \left[\max_{[0,t]} \|z(s)\|^2 + \|A^{1/2}z(t)\|^2 \right]^{1/2}, \quad t \geq r.$$

Integrating over interval $[\tau, t]$ with $\tau \geq r$ and using (18) we obtain that

$$\|A^{1/2}z(t)\|^2 + \int_\tau^t \|Az(\xi)\|^2 d\xi \leq \|A^{1/2}z(\tau)\|^2 + C_T(R)|\varphi^1 - \varphi^2|_C, \quad t \geq \tau \geq r. \quad (28)$$

Now we integrate (28) with respect to τ over $[r, t]$, change the order of integration, and use (18) to get

$$(t-r) \|A^{1/2}z(t)\|^2 + \int_r^t (\xi-r) \|Az(\xi)\|^2 d\xi \leq C_T(R)|\varphi^1 - \varphi^2|_C, \quad t \geq r.$$

¹We refer to the Appendix for a discussion of closed evolutions. Here we only mention that any continuous mapping is closed and a mapping can be closed but not continuous, see examples in [28] and also in [5, Sect.1.1].

Using the expression for \dot{z} from (23) and also the bounds in (18) and (27) we have that

$$\|\dot{z}(t) + Az(t)\|^2 + \|A^{-1/2}\dot{z}(t)\|^2 \leq C_T(R) \left[\|A^{1/2}z(t)\|^2 + |\varphi^1 - \varphi^2|_C \right], \quad t \geq r.$$

This implies (26).

The $\frac{1}{2}$ -Hölder continuity of the evolution semigroup S_t in the norm of \mathcal{CL} follows from (26).

The closedness of S_t for $t \in (0, r]$ easily follows from (18). \square

Remark 3.5 As it follows from (27) we can obtain a $\frac{1}{2}$ -Hölder continuity relation like (26) *for all* $t \geq 0$ if we assume in addition that one of initial data φ^i possesses the property $\dot{\varphi}^i \in L_2(-r, 0; H)$. In this case the argument above leads to the relation

$$\begin{aligned} \|A^{-1/2}\dot{z}(t)\|^2 + \|A^{1/2}z(t)\|^2 + \int_0^t [\|\dot{z}(\tau)\|^2 + \|Az(\tau)\|^2] d\tau \\ \leq C_T(R) \left[\|A^{1/2}(\varphi^1(0) - \varphi^2(0))\| + |\varphi^1 - \varphi^2|_C \right] \end{aligned} \quad (29)$$

for all $t \in [0, T]$ and for all initial data φ^i such that $|\varphi^i|_{\mathcal{CL}} + |\dot{\varphi}^i|_{L^2(-r, 0; H)} \leq R$. Moreover, one can also see that the set

$$\mathcal{CL}_0 = \{\varphi \in \mathcal{CL} : \dot{\varphi} \in L^2(-r, 0; H)\} \quad (30)$$

is forward invariant with respect to S_t . Thus $\varphi \mapsto S_t\varphi$ is a $\frac{1}{2}$ -Hölder continuous mapping for each $t \geq 0$ on the Banach space \mathcal{CL}_0 endowed with the norm $|\varphi|_{\mathcal{CL}_0} = |\varphi|_{\mathcal{CL}} + |\dot{\varphi}|_{L^2(-r, 0; H)}$. Hence the dynamical (in the classical sense, see, e.g., [1, 4, 36]) system (\mathcal{CL}_0, S_t) arises. However we prefer to avoid property $\dot{\varphi} \in L_2(-r, 0; H)$ in the description of the phase space. The point is that our goal is long-time dynamics and it is well-known (see, e.g., [1, 4, 36]) that the existence of limiting objects requires some compactness properties. Unfortunately we cannot guarantee these properties in the space \mathcal{CL}_0 without serious restrictions concerning the delay term. This is why we prefer to use the observation made in [28] concerning closed evolutions.

Remark 3.6 A similar problem as above we have with *time* continuity of evolution operator S_t . It is clear from (12) and (16) that $t \mapsto S_t\varphi$ is continuous for every $\varphi \in \mathcal{CL}$ when $t > r$. To guarantee the continuity $t \mapsto S_t\varphi$ *for all* $t \geq 0$ we need make further restriction² on initial data. The main restriction is a compatibility condition at time $t = 0$. To describe this condition we introduce the following (complete) metric space

$$X \equiv \left\{ \varphi \in C^1([-r, 0]; H_{-1/2}) \cap C([-r, 0]; H) \left| \begin{array}{l} \varphi(0) \in H_{1/2}; \\ \dot{\varphi}(0) + A\varphi(0) + F(\varphi) + G(\varphi(0)) = 0 \end{array} \right. \right\} \quad (31)$$

Here the compatibility condition $\dot{\varphi}(0) + A\varphi(0) + F(\varphi) + G(\varphi(0)) = 0$ is understood as an equality in $H_{-1/2}$. The distance in X is given by the relation

$$\text{dist}_X(\varphi, \psi) = \max_{[-r, 0]} \left\{ \|A^{-1/2}(\dot{\varphi}(\theta) - \dot{\psi}(\theta))\| + \|\varphi(\theta) - \psi(\theta)\| \right\} + \|A^{1/2}(\varphi(0) - \psi(0))\|. \quad (32)$$

²We refer to some discussion in [31, 33] for the related PDE models.

One can see that X is a closed subset in the Banach space \mathcal{CL} and the topology generated by the metric dist_X coincides with the induced topology of \mathcal{CL} (see 9).

In the following assertion we collect several dynamical properties of the evolution semigroup S_t which are direct consequences of Theorem 3.3 and Proposition 3.4 and Remark 3.6.

Proposition 3.7 *Under the conditions of Theorem 3.3 problem (1) generates an evolution semigroup S_t of closed mappings on \mathcal{CL} such that*

- (a) $S_t\mathcal{CL} \subset X$ for every $t \geq r$ and the set $S_t B$ is bounded in X for each $t \geq r$ when B is bounded in the space \mathcal{CL} ;
- (b) the set X is forward invariant: $S_t X \subset X$;
- (c) the mapping $\varphi \mapsto S_t \varphi$ is a $\frac{1}{2}$ -Hölder continuous on \mathcal{CL} (and hence on X) for all $t > r$;
- (d) the trajectories $t \mapsto S_t \varphi$ are continuous for $t > r$ and $\varphi \in \mathcal{CL}$. If $\varphi \in X$, then these trajectories are continuous for all $t \geq 0$.

4 Long time dynamics

This section is central for the whole paper. Here we study long-time dynamics of the delay model generated by (1) and (7). The main result stated below in Theorem 4.2 deals with finite-dimensional global and exponential attractors. We refer to the Appendix for the corresponding definitions and the auxiliary facts which we use in our argument.

We first impose the standard hypotheses (see, e.g., [36]) concerning the nonlinear (non-delayed) sink/source term G .

Assumption 4.1 The nonlinear mapping $G : H_{1/2} \rightarrow H$ has the form $G(u) = \Pi'(u)$. Here $\Pi(u) = \Pi_0(u) + \Pi_1(u)$, where $\Pi_0(u) \geq 0$ is bounded on bounded sets in $D(A^{1/2})$ and $\Pi_1(u)$ satisfies the property

$$\forall \eta > 0 \exists C_\eta > 0 : |\Pi_1(u)| \leq \eta \left(\|A^{\frac{1}{2}} u\|^2 + \Pi_0(u) \right) + C_\eta, \quad u \in H_{1/2}. \quad (33)$$

Moreover, we assume that

- (a) there are constants $\eta \in [0, 1)$, $c_4, c_5 > 0$ such that

$$-\langle u, G(u) \rangle \leq \eta \|A^{1/2} u\|^2 - c_4 \Pi_0(u) + c_5, \quad u \in H_{1/2}; \quad (34)$$

- (b) for every $\tilde{\eta} > 0$ there exists $C_{\tilde{\eta}} > 0$ such that

$$\|u\|^2 \leq C_{\tilde{\eta}} + \tilde{\eta} \left(\|A^{\frac{1}{2}} u\|^2 + \Pi_0(u) \right), \quad u \in H_{1/2}. \quad (35)$$

In the case of parabolic models like (6) examples of functions $g(u)$ such that the corresponding Nemytskii operator satisfies Assumptions 2.1(G) and 4.1 can be found in [1] and [36]. The simplest one is $g(u) = u^3 + a_1 u^2 + a_2 u$ with arbitrary $a_1, a_2 \in \mathbb{R}$ in the case when Ω is a 3D domain.

Our main result is the following assertion.

Theorem 4.2 *Let Assumptions 2.1 and 4.1 be in force. Suppose that S_t is the evolution semigroup generated in \mathcal{CL} by (1) and (7). Then there exists $\ell_0 > 0$ such that this semigroup possesses a compact connected global attractor \mathfrak{A} provided $m_F r < \ell_0$, where r is the delay time and m_F is the linear growth constant for F_0 in H defined by the relation*

$$m_F = \limsup_{\|u\| \rightarrow +\infty} \frac{\|F_0(u)\|}{\|u\|}. \quad (36)$$

Moreover, for every $0 < \beta \leq 1$ and $\alpha < \min\{\beta, 1/2\}$ this attractor belongs to the set

$$D_{\alpha, \beta}^R = \left\{ \varphi \in X \left| \begin{aligned} &|A^{1-\beta} \varphi|_C + |A^{-\beta} \dot{\varphi}|_C + \text{Hold}_\alpha(A^{1-\beta} \varphi) + \text{Hold}_\alpha(A^{-\beta} \dot{\varphi}) \\ &+ \left[\int_{-r}^0 \left(\|A^{1/2} \varphi(\theta)\|^2 + \|\dot{\varphi}(\theta)\|^2 \right) d\theta \right]^{1/2} \leq R \end{aligned} \right. \right\} \quad (37)$$

for some $R = R(\alpha, \beta)$, where the Hölder seminorm $\text{Hold}_\alpha(\psi)$ is given by

$$\text{Hold}_\alpha(\psi) = \sup \left\{ \frac{\|\psi(t_1) - \psi(t_2)\|}{|t_1 - t_2|^\alpha} : t_1 \neq t_2, t_1, t_2 \in [-r, 0] \right\}.$$

Assume in addition that there exist $\gamma, \delta > 0$ such that **(a)** the mapping F_0 is globally Lipschitz from $H_{-\gamma}$ into $H_{-1/2+\delta}$, i.e.,

$$\|F_0(u) - F_0(v)\|_{-1/2+\delta} \leq c \|u - v\|_{-\gamma}, \quad u, v \in H_{-\gamma}; \quad (38)$$

and **(b)** the mapping G is globally Lipschitz from $H_{-\gamma}$ into $H_{-1/2+\delta}$, i.e.,

$$\|G(u) - G(v)\|_{-1/2+\delta} \leq c(R) \|u - v\|_{1/2-\gamma}, \quad u, v \in H_{1/2-\gamma}, \quad \|u\|, \|v\| \leq R. \quad (39)$$

Then

(A) The global attractor \mathfrak{A} has finite fractal dimension.

(B) There exists a fractal exponential attractor $\mathfrak{A}_{\text{exp}}$.

We devote the remaining subsections to the proof of Theorem 4.2.

4.1 Existence of a global attractor

To prove the existence of a global attractor it is sufficient to show that the evolution operator possesses a compact absorbing set. In this case we can apply the standard existence result in the form given in [28] for closed semigroups (see the Appendix for more details).

We start with the existence of a bounded absorbing set.

Proposition 4.3 (Bounded dissipativity) *Assume that $u(t)$ solves (1) and (7) with $\varphi \in \mathcal{CL}$. Then one can find $\ell_0 > 0$ such that for every delay time r such that $m_F r < \ell_0$ the following property holds: there exists R_* such that for every bounded set B in \mathcal{CL} there is t_B such that*

$$\|A^{-1/2}\dot{u}(t)\|^2 + \|A^{1/2}u(t)\|^2 + \int_t^{t+1} [\|\dot{u}(\tau)\|^2 + \|Au(\tau)\|^2] d\tau \leq R_*^2 \quad (40)$$

for all $t \geq t_B$ and for all initial data $\varphi \in B$. This yields that the evolution semigroup S_t is dissipative on both \mathcal{CL} and X provided $m_F r < \ell_0$.

Proof. We use the Lyapunov method to get the result. For this we consider the following functional

$$\tilde{V}(t) \equiv \frac{1}{2} [\|u(t)\|^2 + \|A^{1/2}u(t)\|^2] + \Pi(u(t)) + \frac{\mu}{r} \int_0^r \left\{ \int_{t-s}^t \|\dot{u}(\xi)\|^2 d\xi \right\} ds.$$

defined on strong solutions $u(t)$ for $t \geq r$. The positive parameter μ will be chosen later. We note that the main idea behind inclusion of an additional delay term in \tilde{V} is to find a compensator for the delay term in (1). This idea was already applied in [12] for second order in time models with state-dependent term, see also [8, p.480] and [10] for the case of a flow-plate interaction model which contains a linear constant delay term with the critical spatial regularity. The corresponding compensator is model-dependent.

One can see from (33) that there is $0 < c_0 < 1$ and $c, c_1 > 0$ such that

$$c_0 [\|A^{1/2}u(t)\|^2 + \Pi_0(u(t))] - c \leq \tilde{V}(t) \leq c_1 [\|A^{1/2}u(t)\|^2 + \Pi_0(u(t))] + \mu \int_0^r \|\dot{u}(t-\xi)\|^2 d\xi + c. \quad (41)$$

We consider the time derivative of \tilde{V} along a solution. One can easily check that

$$\begin{aligned} \frac{d}{dt} \tilde{V}(t) &= \langle u(t), \dot{u}(t) \rangle + \langle Au(t), \dot{u}(t) \rangle + \langle G(u(t)), \dot{u}(t) \rangle + \frac{\mu}{r} \int_0^r \{ \|\dot{u}(t)\|^2 - \|\dot{u}(t-s)\|^2 \} ds \\ &= \langle \dot{u}(t) + Au(t) + G(u(t)), \dot{u}(t) \rangle - \langle \dot{u}(t), \dot{u}(t) \rangle + \langle u(t), \dot{u}(t) \rangle + \mu \|\dot{u}(t)\|^2 \\ &\quad - \frac{\mu}{r} \int_0^r \|\dot{u}(t-\xi)\|^2 d\xi \\ &= -\langle F(u_t) - h, \dot{u}(t) \rangle - (1-\mu) \|\dot{u}(t)\|^2 - \frac{\mu}{r} \int_0^r \|\dot{u}(t-\xi)\|^2 d\xi \\ &\quad - \|A^{1/2}u(t)\|^2 - \langle F(u_t) + G(u(t)) - h, u(t) \rangle. \end{aligned}$$

The last terms are due to (1):

$$\langle u(t), \dot{u}(t) \rangle = -\langle Au(t), u(t) \rangle - \langle F(u_t) + G(u(t)) - h, u(t) \rangle.$$

By the definition of m_F in (36) for any number M_F greater than m_F we can find $C(M_F)$ such that

$$\|F(u_t)\| \leq \|F_0(u(t-\eta(u_t)))\| \leq M_F \|u(t-\eta(u_t))\| + C(M_F).$$

Therefore

$$\begin{aligned} \|F(u_t)\| &\leq M_F \|u(t - \eta(u_t)) - u(t)\| + M_F \|u(t)\| + C(M_F) \\ &= M_F \left\| \int_{t-\eta(u_t)}^t \dot{u}(\theta) d\theta \right\| + M_F \|u(t)\| + C(M_F), \end{aligned}$$

and thus

$$\|F(u_t)\| \leq M_F \cdot \left[\|u(t)\| + \int_0^r \|\dot{u}(t - \xi)\| d\xi \right] + C(M_F), \quad t \geq r.$$

Since

$$\int_0^r \|\dot{u}(t - \xi)\| d\xi \leq r^{1/2} \left(\int_0^r \|\dot{u}(t - \xi)\|^2 d\xi \right)^{1/2},$$

we have that

$$\begin{aligned} |\langle F(u_t) - h, \dot{u}(t) \rangle| &\leq \frac{1}{2} \|\dot{u}(t)\|^2 + c_0 \|h\|^2 + c_1 M_F^2 \|u(t)\|^2 \\ &\quad + c_2 M_F^2 r \int_0^r \|\dot{u}(t - \xi)\|^2 d\xi + C(M_F), \quad t \geq r. \end{aligned}$$

In a similar way we also have that

$$|\langle F(u_t) - h, u(t) \rangle| \leq c_1 M_F^2 r \int_0^r \|\dot{u}(t - \xi)\|^2 d\xi + C(M_F)(1 + \|u(t)\|^2).$$

Thus

$$\begin{aligned} |\langle F(u_t) - h, \dot{u}(t) \rangle| + |\langle F(u_t) - h, u(t) \rangle| &\leq \\ &\quad \frac{1}{2} \|\dot{u}(t)\|^2 + c_0 M_F^2 r \int_0^r \|\dot{u}(t - \xi)\|^2 d\xi + c_1(M_F)(1 + \|u(t)\|^2). \end{aligned}$$

The relations in (34) and (35) with small enough $\tilde{\eta} > 0$ (and $\eta \in [0, 1)$) yield

$$c_1(M_F)(1 + \|u\|^2) - \|A^{\frac{1}{2}}u\|^2 - (u, G(u)) \leq -a_0 \left[\|A^{1/2}u\|^2 + \Pi_0(u) \right] + a_1(M_F)$$

for some $a_i > 0$ with a_0 independent of M_F . Thus it follows from the relations above that

$$\begin{aligned} \frac{d}{dt} \tilde{V}(t) &\leq - \left(\frac{1}{2} - \mu \right) \|\dot{u}(t)\|^2 \\ &\quad - a_0 \left[\|A^{1/2}u\|^2 + \Pi_0(u) \right] + a_1(M_F) + \left[-\frac{\mu}{r} + a_2 M_F^2 r \right] \int_0^r \|\dot{u}(t - \xi)\|^2 d\xi \end{aligned}$$

for some a_i . Thus using the right inequality in (41) we arrive at the relation

$$\begin{aligned} \frac{d}{dt} \tilde{V}(t) + \gamma \tilde{V}(t) &\leq - \left(\frac{1}{2} - \mu \right) \|\dot{u}(t)\|^2 - (a_0 - \gamma c_1) \left[\|A^{1/2}u\|^2 + \Pi_0(u) \right] \\ &\quad + \left[-\frac{\mu}{r} + \mu\gamma + a_2 M_F^2 r \right] \int_0^h \|\dot{u}(t - \xi)\|^2 d\xi + a_1(M_F). \end{aligned} \quad (42)$$

Therefore taking $\mu = 1/4$ and fixing $\gamma \leq a_0 c_1^{-1}$ we obtain that

$$\frac{d}{dt} \tilde{V}(t) + \gamma \tilde{V}(t) + \frac{1}{4} \|\dot{u}(t)\|^2 \leq C, \quad t \geq r, \quad (43)$$

provided $\gamma r + 4a_2 M_F^2 r^2 \leq 1$. Thus under the condition $4a_2 m_F^2 r^2 < 1$ we can choose $\gamma \in (0, a_0 c_1^{-1}]$ and $M_F > m_F$ such that (43) holds. In particular we have that

$$\frac{d}{dt} \tilde{V}(t) + \gamma \tilde{V}(t) \leq C, \quad t \geq r,$$

which implies

$$\tilde{V}(t) \leq \tilde{V}(r) e^{-\gamma(t-r)} + \frac{C}{\gamma} (1 - e^{-\gamma(t-r)}), \quad t \geq r, \quad (44)$$

when $m_F r < \ell_0$. Using (41) and (17) we can conclude that $|\tilde{V}(r)| \leq C_B$ for all initial data from a bounded set B in \mathcal{CL} . Hence (see (1)) there exists R such that for every initial data from a bounded set B in \mathcal{CL} we have that

$$\|A^{1/2}u(t)\| + \|A^{-1/2}\dot{u}(t)\| + \|\dot{u}(t) + Au(t)\| \leq R \quad \text{for all } t \geq t_B.$$

Moreover, it follows from (43) that

$$\int_t^{t+1} \|\dot{u}(\tau)\|^2 d\tau \leq C_R \quad \text{for all } t \geq t_B.$$

To get this one should multiply (43) by $e^{\gamma t}$, integrate over $[t, t+1]$ and multiply by $e^{-\gamma t}$. Then ultimate boundedness of $\tilde{V}(t)$ (see (44)) and the relation $1 \leq e^{\gamma(\tau-t)}$ for $\tau \geq t$ give the last estimate.

These relations imply (40) and allow us to complete the proof of Proposition 4.3. \square

Remark 4.4 If the mapping F_0 has sublinear growth in H , i.e., there exists $\beta < 1$ such that

$$\|F_0(u)\| \leq c_1 + c_2 \|u\|^\beta, \quad u \in H,$$

then the linear growth parameter m_F given by (36) is zero. Thus in this case we have no restrictions concerning r in the statement of Proposition 4.3. In particular, this is true in the case of bounded mappings F_0 . Moreover, in the latter case the argument can be simplified substantially (we can use a Lyapunov type function without delay terms). For more details we refer to [31, 33].

We use Proposition 4.3 to obtain the following assertion which means that the evolution semi-group S_t is (ultimately) compact.

Proposition 4.5 (Compact dissipativity) *As in Proposition 4.3 we assume that $m_F r < \ell_0$. Then the evolution operator S_t possesses a compact absorbing set. More precisely, for every $0 < \beta \leq 1$ and $\alpha < \min\{\beta, 1/2\}$ the set $D_{\alpha,\beta}^R$ given by (37) is absorbing for some R . This set $D_{\alpha,\beta}^R$ is compact in X provided $0 < \alpha < \beta \leq 1/2$.*

Proof. We first note that the compactness of $D_{\alpha,\beta}^R$ in $X \subset \mathcal{CL}$ for $0 < \alpha < \beta \leq 1/2$ follows from Arzelà-Ascoli theorem in Banach spaces (see, e.g., [34]).

Now we show that $D_{\alpha,\beta}^R$ is absorbing.

Using the mild form of the problem and also the bound in (10) one can also show that

$$\|A^{1-\delta}u(t)\| + \|A^{-\delta}\dot{u}(t)\| \leq C_R(\delta) \quad \text{for all } t \geq t_B, \quad (45)$$

for every $\delta > 0$, where $u(t)$ is a solution possessing property (40).

Now we consider the difference $u(t_1) - u(t_2)$ with $t_1 > t_2$. Namely, using the mild form we obtain

$$\begin{aligned} \|A^{1-\beta}(u(t_1) - u(t_2))\| &\leq \|A^{1-\beta}(e^{-A(t_1-t_2)} - 1)u(t_2)\| \\ &\quad + \int_{t_2}^{t_1} \|A^{1-\beta}e^{-A(t-\tau)}\| \cdot (\|F(u_\tau)\| + \|G(u(\tau)) - h\|) d\tau. \end{aligned}$$

Since (see [21, Theorem 1.4.3, p.26] for related facts)

$$\|A^{-\alpha}(1 - e^{-At})\| \leq t^\alpha \quad \text{and} \quad \|A^\alpha e^{-At}\| \leq \left(\frac{\alpha}{t}\right)^\alpha e^{-\alpha}$$

for all $t > 0$ and $0 \leq \alpha \leq 1$, we obtain

$$\begin{aligned} \|A^{1-\beta}(u(t_1) - u(t_2))\| &\leq |t_1 - t_2|^\alpha \|A^{1-\beta+\alpha}u(t_2)\| \\ &\quad + c_\beta \int_{t_2}^{t_1} \frac{1}{|t_1 - \tau|^{1-\beta}} [C_{R_*} + c|u_\tau|_C] d\tau. \end{aligned}$$

for $t \geq t_B$. Thus for every $0 < \alpha < \beta \leq 1$ we have

$$\|A^{1-\beta}(u(t_1) - u(t_2))\| \leq C_{R_*} |t_1 - t_2|^\alpha \quad \text{for all } t_i \geq t_B, |t_1 - t_2| \leq 1. \quad (46)$$

Similarly to (27) using (46) with $\beta = 1$ and $\alpha = 1/2$ we have that

$$\begin{aligned} \|F(u_{t_1}) - F(u_{t_2})\| &\leq L_F \left| \int_{t_1 - \eta(u_{t_1})}^{t_2 - \eta(u_{t_2})} \|\dot{u}(\xi)\| d\xi \right| \\ &\leq C_{R_*} [|t_1 - t_2| + |u_{t_1} - u_{t_2}|_C^2]^{1/2} \leq C_{R_*} |t_1 - t_2|^{1/2} \end{aligned}$$

for every $t_1, t_2 \geq t_B \geq r$. Thus from (1) and (46) we obtain

$$\|A^{-\beta}(\dot{u}(t_1) - \dot{u}(t_2))\| \leq C_{R_*} |t_1 - t_2|^\alpha \quad \text{for all } t_i \geq t_B, |t_1 - t_2| \leq 1,$$

for every $0 < \alpha < 1/2$. This implies that the set $D_{\alpha,\beta}^R$ given by (37) is absorbing for some R provided $0 < \beta \leq 1$ and $\alpha < \min\{\beta, 1/2\}$. \square

Proposition 4.5 allow us to apply the result from [28] (see Theorem A.4 in the Appendix) to guarantee the existence of a compact connected global attractor.

4.2 Dimension and exponential attractor

The proof of finite-dimensionality is based on the notion of *quasi-stability* which says that the semigroup is asymptotically contracted up to a homogeneous compact additive term. For the convenience we remind the corresponding abstract result in the Appendix.

We can assume that there exists a *forward invariant* closed absorbing set D_0 which belongs to $D_{\alpha,\beta}^R$ for an appropriate choice of the parameters (see Proposition 4.5). We also note that the restriction of S_t on D_0 is *continuous* in both t and initial data in the topology induced by \mathcal{CL} (see (9)). Thus a dynamical system (S_t, D_0) in the classical (see [1, 4, 19, 36]) sense arises. Therefore we can apply the quasi-stability method developed earlier in [5, 6, 7, 8, 9] for continuous evolution models.

Proposition 4.6 (Quasi-stability) *Let Assumptions 2.1 and 4.1 be in force. Assume that (38) and (39) are valid. Then*

$$\begin{aligned} |S_t \varphi^1 - S_t \varphi^2|_{\mathcal{CL}} &\leq C_R e^{-\lambda_1 t} [\|\varphi^1(0) - \varphi^2(0)\|_{1/2} + |\varphi^1 - \varphi^2|_C] \\ &\quad + C_R \max_{s \in [0, t]} \|A^{1/2-\beta}(u^1(s) - u^2(s))\|, \quad t \geq r, \end{aligned} \quad (47)$$

for every $\varphi^i \in D_0$, where $u^i(t) = (S_t \varphi^i)(\theta)|_{\theta=0}$.

Proof. Using the mild form presentation for $u^i(t)$ and (39) we have that

$$\begin{aligned} \|A^{1/2}(u^1(t) - u^2(t))\| &\leq e^{-\lambda_1 t} \|A^{1/2}(u^1(0) - u^2(0))\| \\ &\quad + \int_0^t \|A^{1-\delta} e^{-A(t-\tau)}\| \cdot \left(C \|A^{-1/2+\delta}[F(u_\tau^1) - F(u_\tau^2)]\| + C_R \|u^1(t) - u^2(t)\|_{1/2-\gamma} \right) d\tau. \end{aligned}$$

As in (27) we also have that

$$\begin{aligned} \|A^{-1/2+\delta}[F(u_t^2) - F(u_t^1)]\| &\leq C \left| \int_{t-\eta(u_t^1)}^{t-\eta(u_t^2)} \|A^{-\beta} \dot{u}_2(\xi)\| d\xi \right| + C \|u_t^2 - u_t^1\|_C \\ &\leq C(R) \max_{\theta \in [-r, 0]} \|u^2(t+\theta) - u^1(t+\theta)\| \end{aligned}$$

for every $t \geq 0$. Therefore

$$\begin{aligned} \|A^{1/2}(u^1(t) - u^2(t))\| &\leq c_1 e^{-\lambda_1 t} [\|A^{1/2}(\varphi^1(0) - \varphi^2(0))\| + |\varphi^1 - \varphi^2|_C] \\ &\quad + c_2(R) \max_{s \in [0, t]} \|A^{1/2-\beta}(u^1(s) - u^2(s))\|, \end{aligned}$$

Using (1), (4) and (11) we also have that

$$\|A^{-1/2}(\dot{u}^1(t) - \dot{u}^2(t))\| \leq C(R) [\|A^{1/2}(u^1(t) - u^2(t))\| + |u_t^2 - u_t^1|_C]$$

Thus

$$\begin{aligned} \|A^{-1/2}(\dot{u}^1(t) - \dot{u}^2(t))\| &\leq c_1 e^{-\lambda_1 t} [\|A^{1/2}(\varphi^1(0) - \varphi^2(0))\| + |\varphi^1 - \varphi^2|_C] \\ &\quad + c_2(R) \max_{s \in [0, t]} \|A^{1/2-\gamma}(u^1(s) - u^2(s))\|, \quad t \geq r. \end{aligned}$$

This completes the proof of Proposition 4.6. \square

In order to prove the finite dimensionality of the attractor \mathfrak{A} we apply Theorem A.6 on the attractor with an appropriate choice of operators and spaces. Indeed, let $T > 0$ be chosen such that $q \equiv C_R e^{-\lambda_1 T} < 1$ where C_R is the constant from (47). We define the Lipschitz mapping

$$K : D_0 \mapsto Z_{[0,T]} \equiv C^1([0,T]; D(A^{-1/2})) \cap C([0,T]; D(A^{1/2}))$$

by the rule $K\varphi = u(t), t \in [0, T]$, with u be the unique solution of (1) and (7) with initial function $\varphi \in D_0$. The seminorm $n_Z(u) \equiv \max_{s \in [0,T]} \|A^{1/2-\beta} u(s)\|$ is compact on $Z_{[0,T]}$ due to the compact imbedding of $Z_{[0,T]}$ into $C([0,T]; D(A^\alpha))$ by the Arzelà-Ascoli theorem (see, e.g., [34]).

If we take

$$Y \equiv \{\varphi \in C^1([-r, 0]; H_{-1/2}) \cap C([-r, 0]; H) \mid \varphi(0) \in H_{1/2}\}$$

equipped with the norm (32) and suppose $V = S_T$, then the (discrete) quasi-stability inequality in (48) is valid on D_0 . Hence we can apply Theorem 3.1.20 [5] (see Theorem A.6) with $V = S_T$, $M = \mathfrak{A}$ and the quasi-stability estimate (47) on the attractor \mathfrak{A} which lies in D_0 . Thus $\dim_f \mathfrak{A}$ is finite (in X and thus in \mathcal{CL}).

To prove the existence of a fractal exponential attractor we first use (48) on the set D_0 and then apply Theorem A.7 to show that there exists a finite-dimensional set $A_\theta \subset D_0$ such that (50) holds. Then as in the standard construction (see, e.g., [15] or [27]) we suppose

$$\mathfrak{A}_{exp} = \overline{\cup\{S_t A_\theta : t \in [0, T]\}}.$$

Since $V = S_T$ it is easy to see that \mathfrak{A}_{exp} is exponentially attracting, see (49) in the Appendix.

Since D_0 is included in the set $D_{\alpha,\beta}^R$ given by (37), we have that $t \mapsto S_t \varphi$ is α -Hölder on D_0 and

$$|S_{t_1} \varphi - S_{t_2} \varphi|_Y \leq C_{D_0} |t_1 - t_2|^\alpha, \quad t_1, t_2 \in [0, T], \quad \varphi \in D_0.$$

Therefore in the standard way (see, e.g., [15] or [27]) we can conclude that \mathfrak{A}_{exp} has finite fractal dimension in Y .

This completes the proof of Theorem 4.2.

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A Appendix

Here, for the convenience of the reader, we remind some results used in our work. For more details we refer to the cited sources.

First we collect some definitions and properties, connected to (closed) evolution semigroups. We start with the following notion which was introduced in [28].

Definition A.1 (Closed semigroup) Let \mathcal{X} be a complete metric space. A **closed semigroup** on \mathcal{X} is a one-parameter family of (nonlinear) operators $S_t : \mathcal{X} \rightarrow \mathcal{X}$ ($t \in \mathbb{R}_+$) (or $t \in \mathbb{N}$) satisfying the conditions

(S.1) $S_0 = Id_{\mathcal{X}}$ -identical operator;

(S.2) $S_{t+\tau} = S_t S_\tau$ for all $t, \tau \in \mathbb{R}_+$;

(S.3) for every $t \in \mathbb{R}_+$ the relations $x_n \rightarrow x$ and $S_t x_n \rightarrow y$ imply that $S_t x = y$.

Assumptions (S.1) and (S.2) are the semigroup properties, while (S.3) says that S_t is a closed (nonlinear) map. We note the operator closeness is a well-known concept in the theory of linear (unbounded) operators. To our best knowledge in the context of evolution operators this notion was appeared in [1] as a (weak) closeness of an evolution (strongly continuous) semigroup (see also [4]).

The following notions are standard in the theory of infinite-dimensional evolution semigroups and dynamical systems (see, e.g., [1, 4, 19, 23, 36]).

Definition A.2 (Dissipativity and compactness) A semigroup S_t is **dissipative** if there is a bounded absorbing set $\mathcal{B}_{abs} \subset \mathcal{X}$. That means for any bounded set $B \subset \mathcal{X}$, there exists $t_0 = t_0(B)$ (the entering time) such that $S_t B \subset \mathcal{B}_{abs}$ for all $t \geq t_0$. A semigroup S_t is **compact** if there is a compact absorbing set.

Definition A.3 (Global attractor) A **global attractor** of an evolution semigroup S_t acting on a complete metric space \mathcal{X} is defined as a bounded closed set $\mathfrak{A} \subset \mathcal{X}$ which is invariant ($S_t \mathfrak{A} = \mathfrak{A}$ for all $t > 0$) and attracting.

We recall ([1, 36]) that a set $\mathcal{K} \subset \mathcal{X}$ is called **attracting** for $S(t)$ if, for any bounded set $B \subset \mathcal{X}$,

$$\lim_{t \rightarrow +\infty} d_{\mathcal{X}}\{S(t)B, \mathcal{K}\} = 0,$$

where $d_{\mathcal{X}}\{A, B\} \equiv \sup_{x \in A} \text{dist}_{\mathcal{X}}(x, B)$ is the Hausdorff semi-distance between bounded sets $A, B \subset \mathcal{X}$.

The following assertion is a reformulation of Corollary 6 [28] which also takes into account the statement of [28, Theorem 2]).

Theorem A.4 (Existence of a global attractor) Assume that $S_t : \mathcal{X} \rightarrow \mathcal{X}$ is a closed semigroup possessing a compact connected absorbing set $\mathcal{K}_{abs} \subset \mathcal{X}$. Then there exists a compact global attractor \mathfrak{A} for S_t . This attractor is a connected set and $\mathfrak{A} = \omega(\mathcal{K}_{abs}) = \bigcap_{t \in \mathbb{R}} \overline{\bigcup_{\tau \geq t} S_\tau \mathcal{K}_{abs}}$.

One of the desired qualitative properties of an attractor is its finite-dimensionality. We remind the following definition.

Definition A.5 [4, 36]). Let $M \subset \mathcal{X}$ be a compact set. Then the **fractal (box-counting) dimension** $\dim_f M$ of M is defined by

$$\dim_f M = \limsup_{\varepsilon \rightarrow 0} \frac{\ln n(M, \varepsilon)}{\ln(1/\varepsilon)},$$

where $n(M, \varepsilon)$ is the minimal number of closed balls of the radius ε which cover the set M .

Our proof of the finite-dimensionality of the global attractor used the following abstract result.

Theorem A.6 ([5, Theorem 3.1.20]). Let Y be a Banach space and M be a bounded closed set in Y . Assume that there exists a mapping $V : M \rightarrow Y$ such that

- (i) $M \subset VM$.
- (ii) There exist a Lipschitz mapping K from M into some Banach space Z and a compact seminorm $n_Z(x)$ on Z such that

$$\|Vv^1 - Vv^2\| \leq \gamma \|v^1 - v^2\| + n_Z(Kv^1 - Kv^2) \quad (48)$$

for any $v^1, v^2 \in M$, where $0 < \gamma < 1$ is a constant. Then M is a compact set in Y of a finite fractal dimension and

$$\dim_f M \leq \left[\ln \frac{2}{1+\gamma} \right]^{-1} \cdot \ln m_Z \left(\frac{4L_K}{1-\gamma} \right),$$

where $L_K > 0$ is the Lipschitz constant for K :

$$\|Kv^1 - Kv^2\|_Z \leq L_K \|v^1 - v^2\|, v^1, v^2 \in M,$$

and $m_Z(R)$ is the maximal number of elements z_i in the ball $\{z \in Z : \|z\|_Z \leq R\}$ possessing the property $n_Z(z_i - z_j) > 1$ when $i \neq j$.

We recall (see [15]) that a compact set $\mathfrak{A}_{\text{exp}} \subset \mathcal{CL}$ is said to be **fractal exponential attractor** for S_t iff $\mathfrak{A}_{\text{exp}}$ is a positively invariant set whose fractal dimension is finite and for every bounded set D there exist positive constants t_D , C_D and γ_D such that

$$\sup_{\varphi \in D} \text{dist}_{\mathcal{CL}}(S_t \varphi, \mathfrak{A}_{\text{exp}}) \leq C_D \cdot e^{-\gamma_D(t-t_D)}, \quad t \geq t_D. \quad (49)$$

For details concerning fractal exponential attractors in the case of continuous semigroups, we refer to [15] and also to the recent survey [27]. We only mention that (i) a global attractor can be non-exponential and (ii) an exponential attractor is not unique and contains the global attractor.

The dimension theorem discussed above pertains to negatively or strictly invariant sets M ($M \subseteq V(M)$). To prove the existence of exponential attractors we need an analog of Theorem A.6 for positively invariant sets. More precisely we need the following assertion which was established in [5] and is a version of the result proved in [7] for metric spaces.

Theorem A.7 *Let $V : M \mapsto M$ be a mapping defined on a closed bounded set M of a Banach space Y . Assume that there exist a Lipschitz mapping K from M into some Banach space Z and a compact seminorm $n_Z(x)$ on Z such that the property in (48) holds. Then for any $\theta \in (\gamma, 1)$ there exists a positively invariant compact set $A_\theta \subset M$ of finite fractal dimension satisfying*

$$\sup \{ \text{dist}(V^k u, A_\theta) : u \in M \} \leq r\theta^k, \quad k = 1, 2, \dots, \quad (50)$$

for some constant $r > 0$. Moreover,

$$\dim_f A_\theta \leq \ln m_Z \left(\frac{2L_K}{\theta - \gamma} \right) \cdot \left[\ln \frac{1}{\theta} \right]^{-1},$$

where we use the same notations as in Theorem A.6.

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